Waveform View of Integrated and Differentiated Walsh–Hadamard Transforms

The Walsh–Hadamard Transform (WHT) is usually viewed algebraically as a matrix transform. An equally useful perspective is to view its columns as **waveforms** and study how simple operators reshape those waveforms.

Let H denote the WHT matrix, I a discrete integration (prefix sum) operator, and D a discrete differentiation operator.

Ordinary WHT

The basis vectors of H are square-wave patterns of increasing sequency:

- DC basis: constant.
- Low sequency: slowly changing square waves.
- High sequency: rapidly alternating square waves.

The transform decomposes a signal into square-wave components rather than sinusoidal components.

Integration Before the WHT

Applying integration generates a new transform

HI.

From the waveform viewpoint, integration converts square waves into piecewise-linear waveforms.

Examples:

Original Walsh waveform	**Integrated waveform**
Constant	Ramp
Square wave	Triangle-like waveform
Alternating ±1	Staircase waveform

Integration:

- smooths discontinuities,
- converts jumps into slopes,
- produces basis vectors with greater continuity.

Repeated integration yields increasingly smooth basis functions:

$H, HI, HI^2, HI^3,...$

giving square-wave, triangular, piecewise-quadratic, and higher-order basis families.

Differentiation Before the WHT

Applying differentiation generates

HD.

Differentiation has the opposite effect:

- plateaus become zero,
- transitions become impulses,
- square waves become sparse spike trains.

Examples:

Original Walsh waveform	**Differentiated waveform**
Constant	Zero
Ramp	Constant
Square wave	Impulses at transitions

The resulting basis vectors become increasingly localized around change points.

Repeated differentiation produces:

$H, HD, HD^2, HD^3,...$

with basis functions becoming progressively more edge- and transition-oriented.

Basis Design Perspective

The key observation is that I and D do not merely preprocess signals; they **change the waveform family used by the transform itself**.

Rather than using only square-wave Walsh bases, one obtains families such as:

- Square-wave bases (H)
- Triangular bases (HI)
- Piecewise-polynomial bases (HI^k)
- Impulse-like bases (HD)
- Higher-order edge bases (HD^k)

while retaining fast computation.

Computational Cost

Since:

- WHT costs O(nlogn),
- Integration costs O(n),
- Differentiation costs O(n),

all transforms

HI^k, HD^k

remain O(nlogn).

Thus simple sparse operators can generate a large family of fast transforms with substantially different basis waveforms, sparsity properties, and signal-matching characteristics, without requiring entirely new transform algorithms.